The University of Winnipeg
Department of Mathematics & Statistics
STAT-4401(3): Official list of possible midterm exam problems
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The midterm exam questions will be chosen from those on this list, which will not be "official" until the word "tentative" is removed from the title.

1. Define what is meant by the probability generating function of a non-negative integer-valued random variable X.

2. Suppose $X_1 \sim \text{Bin}(n_1, p)$ and $X_2 \sim \text{Bin}(n_2, p)$. Show that $X_1 + X_2 \sim \text{Bin}(n_1 + n_2, p)$. Show this using the generating function technique.

3. Let $X_1, \ldots, X_n$ be independent Geometric(p)-distributed random variables. Determine the distribution of $X_1+X_2+\ldots+X_n$. You may use either the generating function or the moment generating function technique.

4. You are given that a random variable X, taking values $n = 0, 1, 2, \ldots,$ has the p.g.f. $g_X(t) = \frac{1}{2}\left(1 - \frac{t}{2}\right)^{-1}$. Obtain $P(X = n)$.

5. Let $X \sim N(0, \sigma^2)$. Show that $E(X^{2n+1}) = 0$ for $n = 0, 1, 2, \ldots,$ $E(X^{2n}) = \frac{(2n)!}{2^n n!}\sigma^{2n}$, for $n = 1, 2, \ldots$ Hint: Use the power series expansion of the moment generating function.

6. Prove the addition theorem for a gamma distribution with the aid of characteristic functions. That is, if $X_1, X_2, \ldots, X_n$ are i.i.d. Gamma(p, a) random variables, then $X_1 + X_2 + \ldots + X_n$ is also Gamma-distributed.

7. Prove the addition theorem for Poisson random variables with the aid of characteristic functions.

8. Let X and Y be independent random variables and suppose that Y is symmetric around zero. Show that XY is symmetric.

9. Let the random variables $Y, X_1, X_2, \ldots,$ be independent and Exp$\left(\frac{1}{a}\right)$-distributed. Suppose that $Y \sim \text{Fs}(p)$, where $0 < p < 1$. Find the distribution of

$$Z = \sum_{j=1}^{Y} X_j.$$

10. Suppose $X_1, X_2, \ldots$ are independent, identically distributed Linnik(α)-distributed random variables, that $N \sim \text{Fs}(p)$ and that N is independent of $X_1, X_2, \ldots$ Show that $p^{1/\alpha}(X_1 + X_2 + \ldots + X_N)$ is again, Linnik(α)-distributed.
Remark. The characteristic function of the Linnik(α) $(\alpha > 0)$ is $\varphi(t) = (1 + |t|^\alpha)^{-1}$.

11. A miner has been trapped in a mine with three doors. One takes him to freedom after one hour, one brings him back to the mine after 3 hours and the third one brings him back after 5 hours. Suppose he wishes to get out of the mine and that he does so by choosing one of the three doors uniformly at random, and continues to do so until he is free. Find the generating function, the mean and the variance for the time it takes him to reach freedom.

12. Let $Z(n)$ be a Galton-Watson branching process where $Z(n)$ denotes the number of individuals born at time n, and $Z(1) = 1$. Let Y be the family size distribution, and let η be the probability of ultimate extinction.

    (a) Suppose that $Y \sim$ Geometric($p = \frac{1}{4}$), so that

    $$P(Y = y) = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^y \quad y = 0, 1, 2, \ldots$$

    Let $G(s) = E(s^Y)$ be the probability generating function of Y. Show that

    $$G(s) = \frac{1}{4 - 3s}, \quad |s| < \frac{4}{3}.$$

    (b) Let $G_2(s)$ be the probability generating function of Z_2. Find $G_2(s)$, and simplify your expression.

    (c) Find the probability of eventual extinction, η.

    (d) Find the probability that the branching process goes extinct *at* generation $n = 3$.

    (e) Now suppose that $Y \sim$ Poisson(0.5), so that

    $$P(Y = y) = \frac{0.5^y}{y!}e^{-0.5}, y = 0, 1, 2, \ldots$$

    Let $G(s) = E(s^Y)$ be the probability generating function of Y. Show that for $\infty < s < \infty$,

    $$G(s) = e^{0.5(s-1)}.$$

    (f) Using $Y \sim$Poisson(0.5), state the probability of eventual extinction, η.

13. Let $Z(n)$ be a Galton-Watson branching process, where $Z(n)$ denotes the number of individuals born at time n, and $Z(0) = 1$. Let Y be the family size distribution, and suppose that $Y \sim$ Binomial(3,0.5).

    (a) Let $G(s) = E(s^Y)$ be the probability generating function of Y. Working from the probability function of Y, show that for $\infty < s < \infty$,

    $$G(s) = \frac{1}{8}(s^3 + 3s^2 + 3s + 1).$$

    (b) Show that $P(Z(2) = 0) = 0.178$

(c) What is the probability that the process is *not* extinct by generation 2?

(d) Find the probability of eventual extinction, η.

(e) Suppose that $Z(2) = 3$. Find the probability of eventual extinction.

14. # 37, p.96 of the textbook.

15. # 43, p.97 of the textbook.

16. # 48, p.98 of the textbook.