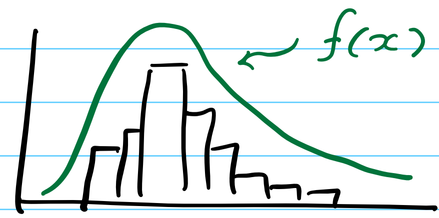
STAT_1302 ; Lecture 12 ; Feb. 15, '24

Test 1 Info. on Nexus; Ass't #3 on Nexus.

Ch. 11. Chi_Square Tests

§11.1 The Chi-square Distribution

Data:



The Chi-square distribution is a probability distribution that models some continuous random variables (e.g. the test statistics in Ch. 11) that is skewed to the right and it is for r.v.'s that are positive.

$f(x)$ (the Chi-square probability density function) smoothes the histogram. Its behaviour is determined by a degrees of freedom parameter (ν).

Read ν as "nu". $\nu = 1, 2, 3, \ldots$

Remark: The entire area under the prob. density function of a Chi-square distribution equals one.

Write $X \sim \chi^2_\nu$ and read this a the r.v. X follows a Chi-square distribution with ν degrees of freedom.

* The mean, variance and Shape of a Chi-square distribution are determined by the degrees of freedom, ν.

Table VI (App. B) provides the upper α points of the χ^2_ν-distribution.

Ex. $X \sim \chi^2_4$. What is $P(X > 7.779)$?

Ans. 0.100

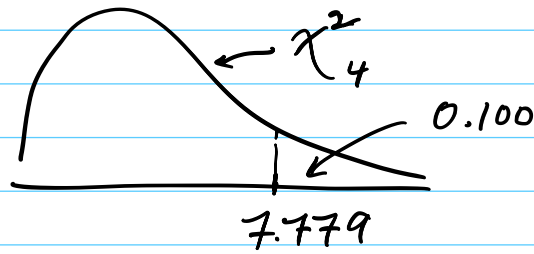


Table VI

df	Area in Right Tail 0.100
	⋮
4	7.779

§ 11.4 Inferences about the Population Variance

Result: Let $X_1, \ldots, X_n$ be a random from a normal population with mean μ and standard deviation σ. And $\frac{n}{N} < 0.05$ (i.e. $X_1, \ldots, X_n$ are independent random variables). Then

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$$

follows a Chi-square distribution with $n-1$ degrees of freedom.

A $100(1-\alpha)\%$ CI for σ^2 where $X_1, \ldots, X_n$ is a random Sample from a normal population with $\frac{n}{N} < 0.05$:

df.

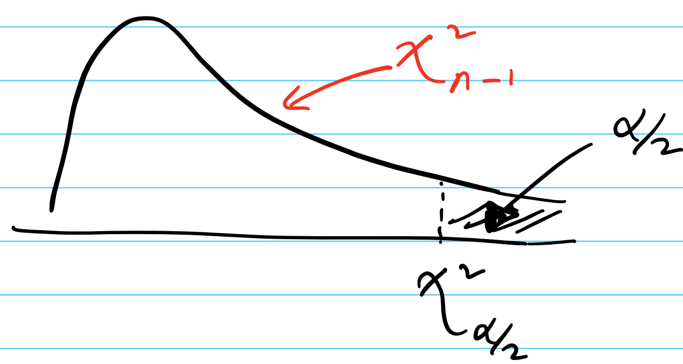
$$\left(\frac{(n-1)S^2}{\chi^2_{\alpha/2}} , \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}} \right)$$

where

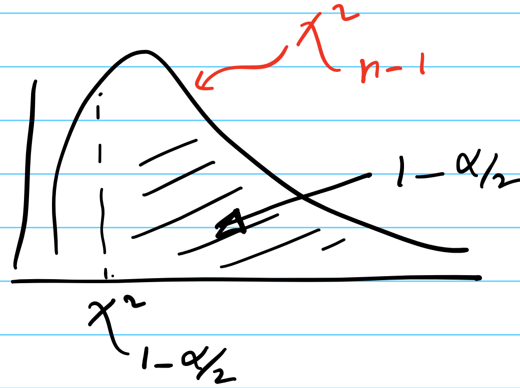
n is the Sample Size;

S^2 " " " variance

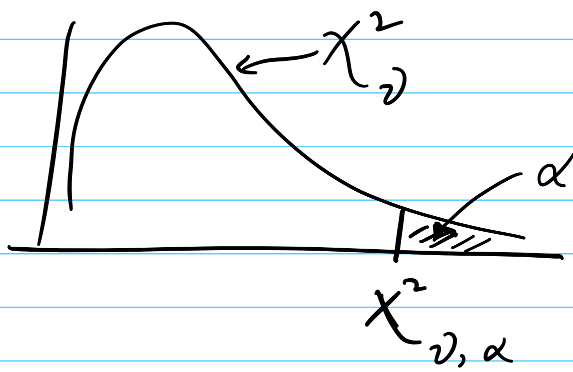
$\chi^2_{\alpha/2}$:



$\chi^2_{1-\alpha/2}$



In general, the notation $\chi^2_{\nu,\alpha}$ means



Ex. A Prof's "50-minutes" lecture varies in length. The prof. claims that the variance of her lectures is within 2 minutes^2. A random sample of 23 of these lectures was timed giving a sample variance of 2.7 minutes^2. Compute a 98% confidence interval for the variance and the standard deviation of all 50-minutes lectures given by this Prof. State any assumptions you are making.

Assumptions: Lecture lengths are normally distributed and independent of each other (or, write $\frac{n}{N} < 0.05$).

$$\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2}} , \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}} \right)$$

$n = 23$; $s^2 = 2.7$; $1 - \alpha = 0.98$; $\alpha = 0.02$;

$\alpha/2 = 0.01$; $1 - \alpha/2 = 1 - 0.01 = 0.99$

$$\left.\begin{aligned} \chi^2_{\alpha/2} &= \chi^2_{0.01} = 40.289 \\ \chi^2_{1-\alpha/2} &= \chi^2_{0.99} = 9.542 \end{aligned}\right\} \begin{aligned} &\text{See Table VI} \\ &df. = n-1 = 22. \end{aligned}$$

$$\left(\frac{22 \times 2.7}{40.289} , \frac{22 \times 2.7}{9.542} \right)$$

$= (1.474, 6.225)$ is the 98% CI for σ^2.

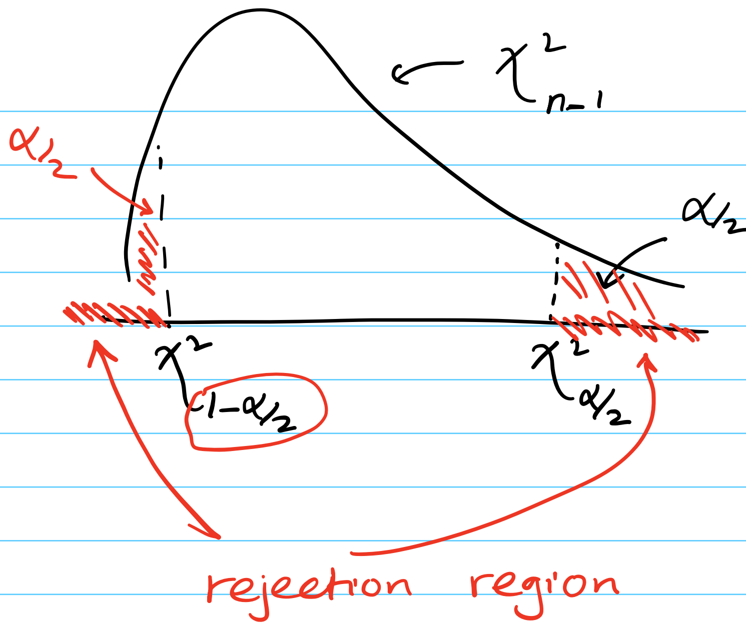
A 98% CI for σ is

$$\left(\sqrt{1.474}, \sqrt{6.225}\right) = (1.214, 2.495)$$

Interpretation: We are 98% Confident that the variance of the lectures varies between 1.474 and 6.225 minutes^2 while the Standard deviation of the lectures varies between 1.214 and 2.495 minutes.

Hypothesis Testing for σ^2:

Test	Rejection Region
$H_0: \sigma^2 = \sigma_0^2$, $H_1: \sigma^2 > \sigma_0^2$	$\chi^2 > \chi^2_{\alpha}$
$H_0: \sigma^2 = \sigma_0^2$, $H_1: \sigma^2 < \sigma_0^2$	$\chi^2 < \chi^2_{1-\alpha}$
** $H_0: \sigma^2 = \sigma_0^2$, $H_1: \sigma^2 \neq \sigma_0^2$	$\chi^2 > \chi^2_{\alpha/2}$ or $\chi^2 < \chi^2_{1-\alpha/2}$



** Can use the $100(1-\alpha)\%$ CI for σ^2 to test $H_0: \sigma^2 = \sigma_0^2$ vs. $H_1: \sigma^2 \neq \sigma_0^2$ by checking to see if σ_0^2 is in the CI. If $\boxed{\text{yes}}$, we fail to reject H_0. If $\boxed{\text{no}}$, we reject H_0.

Test Statistic:

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2} \leftarrow \text{Given}$$

Under H_0, $$\chi^2 = \frac{\overbrace{(n-1)}S^2}{\sigma_0^2} \sim \chi^2_{\underbrace{n-1}}$$

P-value approach: not tested.

Ex. A bank manger does not want the variance of waiting times to be greater than 4 minutes^2. A random sample of 25 customers gave a variance of waiting times to be 8.3 minutes^2. Assume waiting times are normally distributed. Is the waiting time for all customers at this bank more than 4 minutes^2. Let $\alpha = 0.01$.

Sol'n:

Let σ^2 be the pop. variance of waiting times.

$H_0: \sigma^2 = 4$

$\alpha = 0.01$

$H_1: \sigma^2 > 4$

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(25-1)\,8.3}{4} = 49.8$$

Reject H_0 if $\chi^2 > \chi^2_{0.01}$; $df = 25-1 = 24$.

$$\chi^2_{24;0.01} = 42.980.$$

Table VI

	Area in the Right Tail
df	0.01
⋮	⋮
24	$\boxed{42.980}$

Reject H_0 if $\chi^2 > 42.980$.

Since $\chi^2 = 49.8 > 42.980$, we reject H_0.

Conclusion: We are 99% Confident that the variance of waiting times at this bank is more than 4 minutes^2.