Objectives:

- Binomial Distribution and it's moments: Section 5-4 of text.

- In this section we study a discrete probability distribution known as the Binomial probability distribution.

- Consider a random variable Y, that takes on one of two possible values, say, 0 (failure) or 1 (success).

- And, Y has the following probability distribution

y	0	1
$P(y)$	1-p	p

  We say, Y is a Bernoulli random variable or Bernoulli trial with success probability p and write

$$Y \sim Bernoulli(p).$$

- Experiment: Flip a biased coin where Heads once and observe the outcome. Suppose Heads is expected to appear 60% of the time.

- Random variable: Outcome of flipping this ‘unfair’ coin once.

- Possible Outcomes: Heads, Tails.

- Call observing Heads as a success (S). Call observing Tails a failure (F).

- $P(S) = 0.6 = p$, $P(F) = 0.4 = 1 - p = q$.

- $X \sim Bernoulli(0.6)$.

Definition

A **binomial probability distribution** results from a procedure that meets all of the following requirements:

1. The procedure has a fixed number of trials.
2. The trials must be independent. That is, the outcome of any individual trial doesn't affect the probabilities in the other trials.
3. Each trial must have all outcomes classified into two categories.i.e. each trial is a Bernoulli random variable.
4. The probabilities must remain constant for each trial.

Notation for Binomial Probability Distributions

S and **F** (success and failure) denote the two possible categories of all outcomes.

- $P(S) = p$; p is the probability of a success
- $P(F) = 1 - p = q$; q is the probability of a failure
- n: the number of fixed trials
- x: the observed number of successes in n trials; x can take values $0, 1, \ldots, n$

Definition of a Binomial Random Variable

A random variable that is the number of successes in n independent Bernoulli trials with probability of success p on each trial is called a **binomial** random variable with *parameters n and p*. We write

$$X \sim Bin(n, p).$$

Example 2

Let X, be the random variable that counts the number of girls born to a couple planning to have four children. Suppose that the probability of observing a boy is **0.53**. Is X a binomial random variable?

We need to check if all the requirements of a binomial random variable are met.

1. X counts the number of successes (girls) out of four trials.
2. Only two possible values are possible for each outcome. The trials are Bernoulli trials.
3. Trials are independent; the outcome of each birth is not affected by the outcome of any other birth.
4. $P(\text{"}Success\text{"}) = 0.47$ for all trials.

Requirements of Binomial experiment are met $\Rightarrow X \sim Bin(4, 0.47)$.

Example 3

Genetics says that children receive genes from their parents independently. Each child has probability **0.25** of having blood type O. If a set of parents has **5** children, does X, the random variable which counts the number of children with blood type O, follow a binomial distribution?

We need to check if all the requirements of a binomial random variable are met.

Chapter 5: Discrete Probability Distributions

Section 5-4: Binomial Probability Distributions; Example 3 Cont'd

1. X counts the number of successes (blood type O children) out of five trials.
2. Only two possible values are possible for each outcome if we view the outcomes as "type O" and "not type O".
3. Trials are independent; the outcome of each child's blood type is not affected by the outcome of any of the other children's blood type (biologically plausible??).
4. $P(\text{"}Success\text{"}) = 0.25$ for all trials.

$$X \sim Bin(5, 0.25).$$

When sampling without replacement, the events can be treated as if they were independent if the sample size is small relative to the population size, i.e. $\frac{n}{N} < 0.05$.

Example 4

A company inspects a random sample of **10** empty plastic containers from a shipment of **10,000** and examines them for traces of benzene. Suppose **10%** of the containers have traces of benzene. Is, X, the number of containers contaminated with benzene a binomial random variable?

We need to check if all the requirements of a binomial random variable are met.

- Technically, $P(Success) = P(\text{a container with benzene})$ is not 10% for all 10 containers due to sampling without replacement.
- However, the sampling fraction is $10/10,000 = 0.001 < 0.05$.
- Therefore, since the sampling fraction is less than 5%, we may view the events as independent Bernoulli trials with $P(Success) = 0.1$.
- That is, $X \sim Bin(10, 0.1)$.

Let $X \sim Bin(n, p)$. We find $P(x)$, the probability that X takes the value x using the following.

The Binomial Probability Formula

$$P(x) = {}_nC_x\, p^x q^{n-x}, \text{for } \; x = 0, 1, \ldots, n$$

Recall: $q = 1 - p$.

$${}_nC_x = \binom{n}{x} = \frac{n!}{(n-x)!x!}, \quad \text{where} \quad n! = n \times (n-1) \times \ldots \times 1.$$

For example, $4! = 4 \times 3 \times 2 \times 1 = 24$ and

$$ {}_4C_2 = \binom{4}{2} = \frac{4!}{(4-2)!\,2!} = \frac{4!}{2! \times 2!} = \frac{4 \times 3 \times 2!}{2! \times 2!} = 6. $$

Some facts:

$0! = 1$ and $1! = 1$

$n!$ is read as n factorial.

Example 5

Let X, be the random variable that counts the number of girls born to a couple planning to have four children. Suppose that the probability of observing a boy is **0.5**. Verify the probability distribution of X is the following:

x	0	1	2	3	4
$P(x)$	1/16	1/4	3/8	1/4	1/16

Example 5

Let X, be the random variable that counts the number of girls born to a couple planning to have four children. Suppose that the probability of observing a boy is **0.5**. Verify the probability distribution of X is the following:

x	0	1	2	3	4
$P(x)$	1/16	1/4	3/8	1/4	1/16

$$X \sim Bin(4, 0.5).$$

Therefore, $P(2) = {}_4C_2\, 0.5^2(1-0.5)^{4-2} = \dfrac{3}{8}$.

Example 4 Revisited:

A company inspects a random sample of **10** empty plastic containers from a shipment of **10,000** and examines them for traces of benzene. Suppose **10%** of the containers have traces of benzene. What is the probability that the sample contains at most one contaminated container?

Example 4 Revisited:

$$\begin{aligned} P(X \leq 1) &= P(X = 1, \text{or} X = 0) \\ &= P(0) + P(1) \\ &= {}_{10}C_0\, 0.1^0 (1 - 0.1)^{10-0} + {}_{10}C_1\, 0.1^1 (1 - 0.1)^{10-1} \\ &= 0.7361. \end{aligned}$$

Example 5

When a survey calls residential telephone numbers at random, **80%** of calls fail to reach a live person. A random dialling machine makes **15** calls.

(a) What is the probability that exactly three calls reach a person?

(b) What is probability that at least one call reaches a person?

Example 5 Revisited:

- Put $X =$ the number of live calls out of 15.
- Then $X \sim Bin(15, 0.2)$.
- $P(3) = {}_{15}C_3\, 0.2^3 0.8^{12} = 0.250$.
- $P(X \geq 1) = P(1) + P(2) + \ldots + P(15)$. Too much work!
- $P(X \geq 1) = 1 - P(X < 1) = 1 - P(0) = 1 - 0.8^{15} = 0.965$.

Suppose $X \sim Bin(n, p)$.

Mean

$$\mu = E(X) = np$$

Variance

$$\sigma^2 = npq = np(1-p)$$

Standard Deviation

$$\sigma = \sqrt{npq} = \sqrt{np(1-p)}$$

These expressions are given on the final exam.

Example 4 Revisited:

A company inspects a random sample of **10** empty plastic containers from a shipment of **10,000** and examines them for traces of benzene. Suppose **10%** of the containers have traces of benzene. Find the mean and standard deviation of the number of containers out of a sample of ten with benzene.

Example 4 Revisited:

- $X \sim Bin(10, 0.1)$.
- Mean: $\mu = np = 10(0.1) = 1$
- Standard Deviation: $\sigma = \sqrt{npq} = \sqrt{10(0.1)(0.9)} = 0.9487$